JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 6, November–December 1994

Engineering Notes

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Covariance Control Parameterization for Combined Optimization of Structures and Controllers

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Introduction

THE idea that one can improve the overall system performance of an actively controlled structure by simultaneously optimizing the structural design and the control design is an attractive concept.[1,2] However, one cannot be sure that the true potential of combined structure-control design can be realized by control optimization methods that do not directly address the multiobjective nature of the control/structure problem. The purpose of this Note is to present and investigate a parameterization of control laws amenable to a numerical, multiobjective optimization of both the structure and the controller when the controller must satisfy simultaneous, often conflicting side constraints on performance, stability, and robustness.

The method presented and disucssed below is based on a control design methodology developed in Refs. 3 and 4. This method was originally intended to achieve a control design by picking the closed-loop system covariance matrix. We build on these previous results by using the spectral decomposition of the covariance matrix as the design parameters of the controller within a numerical optimization program. By an example application, we show how this parameterization is a reasonable choice for combined numerical optimization studies, because it is analytically differentiable and the design constraints are differentiable functions of the design parameters.

Covariance Control Theory

In this section, we present an overview of covariance control theory as presented in Refs. 3 and 4. Consider the following structural dynamic model of order n_x:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + D_p w \\ z &= M_p x_p \\ y &= C_p x_p \end{aligned} \tag{1}$$

where x_p is the system state vector, u is the vector of actuator forces, w is the vector of disturbances, z is the vector representing the output of the sensors, and y is the vector of controlled outputs.

Received Dec. 21, 1992; revision received Feb 1, 1994; accepted for publication Feb. 20, 1994.

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Any continuous, linear feedback controller of order n_c can be modeled by

$$\dot{x}_c = A_c x_c + F z \qquad u = K x_c + H z \tag{2}$$

where x_c is the controller state vector. The design parameters defining the control system are, generally, the entries of the A_c, F, K, and H controller matrices. These can be grouped into a single matrix G of the feedback gains:

$$G = \begin{bmatrix} H & K \\ F & A_c \end{bmatrix} \tag{3}$$

Provided the plant is observable and controllable, stability is based on finding a Lyapunov function for the linear closed-loop system. This amounts to finding a positive-definite solution X to the following:

$$(A + BGM)X + X(A + BGM)^T + DWD^T + X_0 = 0 \tag{4}$$

where X is the "one-at-a-time" closed-loop system covariance matrix,[3] which includes as states both the plant states x_p and the controller states x_c:

$$x = \begin{bmatrix} x_p \\ x_c \end{bmatrix} \tag{5}$$

so that the partitions of X are

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix} \tag{6}$$

where X_p is the plant covariance, X_c is the controller covariance, and X_{pc} is the coupling covariance between the plant and the controller.

A parameterization of control laws can be developed by realizing that G can be thought of as a function of X. To do this, we can *analytically* solve Eq. (4) for G as a function of X. This "X parameterization," originally described in Ref. 3, explicitly characterizes all reduced-order linear stabilizing controllers that assign the covariance matrix X to the closed-loop system. All reduced-order, stabilizing control laws can, in this manner, be parameterized in terms of the closed-loop covariance by the following closed-form equation relating X to G:

$$\begin{aligned} G &= -\tfrac{1}{2}B^+Q(2I - BB^+)X^{-1}M^+ \\ &\quad + \tfrac{1}{2}B^+(\psi^T - \psi)BB^+X^{-1}M^+ \\ &\quad + B^+XM^+M(I - \Gamma\Gamma^+)S(I - \Gamma\Gamma^+)M^+ \\ &= G_1 + G_2 S G_3 \end{aligned} \tag{7}$$

in which

$$\begin{aligned} Q &= XA^T + AX + DWD^T + X_0 \\ \psi &= L^+(I - M^+M)X^{-1}Q(2I - L^+L) + QL^+L \\ L &= (I - M^+M)X^{-1}BB^+ \\ \Gamma &= MM^+X(I - BB^+) \end{aligned} \tag{8}$$

where the superscript plus denotes the Moore-Penrose inverse. The skew-symmetric matrix S in Eq. (7) can be uniquely defined as a function of X to minimize the control effort.[4] In the example examined in this Note as well as a number of other examples, it was found that S has a small influence on G, so it will be dropped from the subsequent development. The particular G obtained from Eq. (7) is said to "assign" X to the closed-loop system. The control design problem then consists of finding an X that meets all the design objectives.

It should be noted that the order of the controller is explicit in the dimensions chosen for the assigned X. This means that the resulting controller will stably interact with modes contained within the model, even though the model order and controller order are not the same. This is an important feature of any practical structural control design methodology.

For G to assign X, two equality constraints must be satisfied:

$$C_1(X_p) = (I - B_p B_p^+) Q_p (I - B_p B_p^+) = 0$$
$$C_2(X) = (I - M_p^+ M_p) \bar{Q}_p (I - M_p^+ M_p) = 0 \tag{9}$$

in which

$$Q_p = X_p A_p^T + A_p X_p + D_p W D_p^T + X_{0p}$$
$$\bar{Q}_p = \bar{X}_p^{-1} \bar{F} \bar{X}_p^{-1}$$
$$\bar{X}_p = X_p - X_{pc} X_c^{-1} X_{pc}^T$$
$$\bar{F} = \bar{X}_p A_p^T + A_p \bar{X}_p + D_p W D_p^T + X_{0p} + X_{pc} X_c^{-1} X_{0c} X_c^{-1} X_{pc}^T \tag{10}$$

These "assignability conditions" $C_1(X_p)$ and $C_2(X)$ serve the role of the controllability and observability constraints implicit in the linear quadratic Gaussian (LQG) formulation. Although it appears that these are highly nonlinear conditions, in fact they are relatively weak: The space they define is, from our experience, very large and smooth, apparently because they define a relatively small null-space for the submatrices of the covariance matrix.

Control Parameterization Using the Covariance Matrix

The difficulty of using covariance control theory in practice is the transformation of the performance and robustness constraints into an assignable covariance matrix X. We sought instead to use the covariance matrix itself as the design parameters of the controller within a numerical optimization procedure. If this is successful, the task of finding a control law would be accomplished by a numerical search within a more global system optimization.

The main challenge to using the covariance matrix as the control design parameters is the fact that closed-loop stability is a constraint that X is positive definite. Rather than impose a nonlinear side constraint on X for positive definiteness, we further parameterized X so that the stability constraint is more natural in the design parameters. In particular, we considered a spectral decomposition of X. The design variables for the optimization then become the eigenvalues and eigenvector elements of X. Stability is then a simple constraint that the eigenvalues of X be greater than zero.

An advantage of parameterizing the controller using X is that many performance measures can be easily written in terms of X. Specifically, output saturation, sensor saturation, and actuator saturation can be very easily written in terms of X.[6] In addition, control effort, robustness (stability and performance), and pointing accuracy can be written in terms of X.[6]

Nonlinear Covariance Optimization Problem

In summary, the following constrained optimization problem was formulated using the X parameterization:

Minimize control effort V:

$$V = E_\infty(\bar{u}^T R \bar{u}) = \mathrm{tr}\big(H M_p X_p M_p^T H^T R + H M_p X_{pc} K^T R + K X_{pc}^T M_p^T H^T R + K X_c K^T R\big)$$

Subject to

$$\sigma_i(y) < \bar{\sigma}_i \qquad C_1(X) = 0 \qquad C_2(X) = 0$$
$$i = 1, \ldots, n_y \tag{11}$$

The above nonlinear programming problem can be solved using a number of well-known optimization programs. Since we were interested in ultimately using a multiobjective nonlinear search that combines structural and control objectives and constraints, we solved this problem using the KSOPT Pareto-optimal search program.[5] The algorithm used in KSOPT is further described in the next several paragraphs.

Given a typical multiobjective constrained optimization problem with inequality constraints, KSOPT, using the Kreisselmeier-Steinhauser (KS) function,[5] creates a single surface combining all of the constraint boundary surfaces and the objective function(s) surface(s). The optimization algorithm minimizes the KS function, which is defined by

$$\mathrm{KS}(\hat{X}) = f_{\max} + \frac{1}{\rho} \ln \sum_{k=1}^{K_f} \exp[\rho f_k(\hat{X}) - f_{\max}] \tag{12}$$

where

$$K_f = N_{\mathrm{obj}} + N_{\mathrm{con}} \tag{13}$$

$$f_k = \begin{cases} F_k & k = 1, \ldots, N_{\mathrm{obj}} \\ g_k & k = N_{\mathrm{obj}} + 1, \ldots, N_{\mathrm{obj}} + N_{\mathrm{con}} \end{cases} \tag{14}$$

where $f_{\max}$ is the maximum value of the K_f functions, N_{obj} is the number of objectives, N_{con} is the number of constraints, $\hat{X}$ is the vector of design variables, and ρ is a weighting parameter that is varied by the numerical algorithm to aid the Pareto-optimal convergence.

The use of the KS function for converting a constrained minimization problem to an unconstrained one transforms the objective function(s) into goal constraint(s). These constraints are then included in the KS function along with the inequality constraints. As KSOPT searches for the minimum value of the KS function, the design point is moved toward a compromise minimum. In representing the objective function(s) as goal constraint(s), the constrained compromise minimum satisfies the classical definition of a Pareto-optimal point. In Ref. 5 the relationship between this method and that of goal programming is shown, proving that it is capable of finding a Pareto-optimal point. The algorithm belongs in the class of sequential unconstrained minimization techniques (SUMT).

We should also note that our implementation used analytically evaluated gradients of the cost functions and constraints. Because of the complexity of the resulting equations, however, these expressions are not presented here. Instead, the reader is referred to Ref. 6.

Example Application: Four-Mass Oscillator

To demonstrate the convergence of the covariance optimization search, we present results for a simple, four-mass oscillator. This example was also studied in Ref. 7. The problem is a noncollocated sensor/actuator one with an actuator on the first mass and a sensor on the last mass. The controller uses velocity feedback and the performance output is the displacement of the last mass. The system order is 8 and we constrained the controller order to be 6. We requested a 10% reduction in the displacement of the last mass below the open-loop sensitivity.

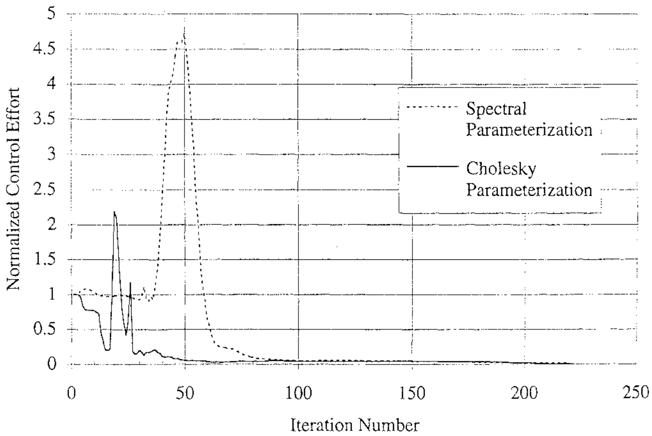

Fig. 1 Normalized control effort vs iteration number for four-mass problem.

Table 1 Summary of results for four-mass oscillator

	Control effort reduction	Output cost reduction	C_1 norm	C_2 norm	Number of iterations
Spectral decomposition	5128%	21%	2.8×10^{-3}	4.4×10^{-2}	220

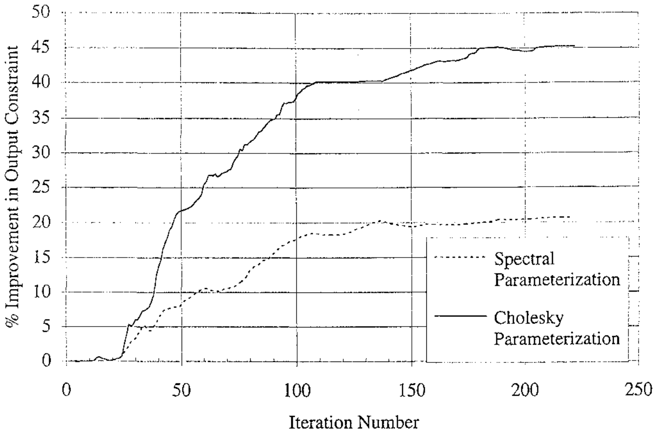

Fig. 2 Percentage reduction in output constraint vs iteration number for four-mass problem.

Using our parameterization of the controller, there are 210 design variables and 178 constraints. The results are shown in Figs. 1 and 2. Figure 1 plots the normalized control effort vs iteration number and Fig. 2 plots the percentage reduction in the output singular value.

The optimizer quickly found a design that satisfied the requested improvement of 10%. It achieved this in 57 iterations where one iteration is one unconstrained minimization. However, we allowed the algorithm to continue to search beyond the requested performance constraint. This resulted in a significant improvement in both the output constraint and the control effort, as summarized in Table 1.

Conclusions

A new parameterization of controllers for application to multiobjective optimizations has been developed and investigated. In this method, the spectral decomposition of the closed-loop covariance matrix is used as the design variables for the controller. This parameterization allows stability to be expressed as a simple side constraint on the design parameters, and performance and robustness constraints can be expressed as differentiable functions of the design parameters without the need for an intervening Lyapunov or eigenproblem solution.

Acknowledgments

This work was supported in part by National Science Foundation Grant MSS-9007276. Also, the first author was supported in part by a David Ross Graduate Research Fellowship while preparing his Ph.D. dissertation.

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Optimization for Efficient Structure-Control Systems

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Introduction

REFERENCE 1 introduced the concept of power efficiency of structure-control systems (SCSs) and presented analyses of several linear quadratic regulator designs on the basis of their efficiencies. Encouraged by the results of Ref. 1, Ref. 2 introduced an efficiency modal analysis of an SCS. The efficiency modal analysis leads to identification of controller efficiency modes distinct from the structural natural modes. Two types of efficiency are defined in Ref. 1 for the SCS: global efficiency and the relative model efficiency. In this Note the focus is on the relative model efficiency. The model efficiency compares the control power expended on a reduced-order model to the total power expenditure on a full-order truth model (TM) via spatially discrete control inputs. In case of finite element models (FEMs) of structural systems the full-order system is the high-dimensional first-cut model of the system known as the Nth-order TM, where N is the total FEM structural degrees of freedom. In the case of distributed-parameter partial differential equation formulation, the full-order model is the ∞-dimensional system. A key feature in controlling a reduced-order model of a high-dimensional (or an ∞-dimensional distributed-parameter) structural dynamic system must be to achieve high power efficiency of the control system while satisfying the control objectives and/or constraints. Formally, this can be achieved by designing the control system and structural parameters simultaneously within an optimization framework. The subject of this Note is to present such a design procedure. Further details on the material presented here can be found in Refs. 3 and 4.

Efficiency Analysis for Structure-Control System

Consider an Nth-order FEM TM of the structural system

$$M\ddot{q} + Kq = DF(t) \tag{1}$$

Received July 23, 1992; revision received May 15, 1994; accepted for publication May 17, 1994.

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